Exact Resultants for Corner-cut Unmixed Multivariate Polynomial Systems using the Dixon Formulation *

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Abstract

Structural conditions on the support of a multivariate polynomial system are developed for which the Dixon-based resultant methods compute exact resultants. For cases when this cannot be done, an upper bound on the degree of the extraneous factor in the projection operator can be determined a priori, thus resulting in quick identification of the extraneous factor in the projection operator. (For the bivariate case, the degree of the extraneous factor in a projection operator can be determined a priori.)

The concepts of a *corner-cut support* and *almost corner-cut support* of an unmixed polynomial system are introduced. For generic unmixed polynomial systems with corner-cut and almost corner-cut supports, the Dixon based methods can be used to compute their resultants exactly. These structural conditions on supports are based on analyzing how such supports differ from box supports of n-degree systems for which the Dixon formulation is known to compute the resultants exactly. Such an analysis also gives a sharper bound on the complexity of resultant computation using the Dixon formulation in terms of the support and the mixed volume of the Newton polytope of the support.

These results are a direct generalization of the authors' results on bivariate systems including the results of Zhang and Goldman as well as of Chionh for generic unmixed bivariate polynomial systems with corner-cut supports.

KEYWORDS: Resultant, Dixon Method, Extraneous Factor, BKK Bound, Support, Support-interior, Corner-Cut, n-degree Systems.

*This research is supported in part by NSF grant nos. CCR-0203051, CDA-9503064 and a grant from the Computer Science Research Institute at Sandia National Labs.

1. Introduction

Resultant matrices based on the Dixon formulation have turned out to be quite efficient in practice for simultaneously eliminating many variables on a variety of examples from different application domains; for details and comparison with other resultant formulations and elimination methods, see [Kapur and Saxena, 1995, Chtcherba and Kapur, 2002e] and `http://www.cs.unm.edu/~artas`. Necessary conditions can be derived on parameters in a problem formulation under which the associated polynomial system has a solution.

A main limitation of matrix-based approach for computing resultants is that often an extraneous factor is generated [Kapur and Saxena, 1997] with no relation with the resultant of a given polynomial system. This paper reports results about polynomial systems for which the Dixon formulation leads to the exact resultant (without any extraneous factor). The concepts of a *corner-cut support* and *almost corner-cut support* of unmixed polynomial systems are introduced based on how such a support deviates from the support of the associated n-degree system whose resultant can be computed exactly using the Dixon formulation. It is proved that for generic unmixed polynomial systems with corner-cut supports and almost corner-cut supports, the Dixon based resultant methods can compute their resultants exactly. These results generalize the earlier results of the authors for bivariate polynomial systems [Chtcherba and Kapur, 2002e] as well as the results of [Chionh, 2001] and [Zhang and Goldman, 2000] on corner-cut supports for bivariate polynomial systems.

An upper bound on the the degree of the extraneous factor can be determined a priori, thus making it easier to identify the resultant in a projection operator. This approach has the distinct advantage of generalizing the most known cases of unmixed polynomial systems (such as n-degree systems as well as systems with corner-cut supports) for which the Dixon formulation is known to compute the resultant exactly [Chtcherba and Kapur, 2000a, 2002e].

The paper bring together the results of [Chtcherba and Kapur, 2002c,d] and builds on the results proved in [Chtcherba and Kapur, 2002c] in which it is shown that the degree of the projection operator computed using Dixon resultant formulations is determined solely by the support hull of the support of a polynomial system. This relationship further generalizes the results in [Chtcherba and Kapur, 2002d] and provides insight into the construction of Dixon matrices. Approximations to an upper bound on the size of the Dixon matrix and hence, the degree of a projections operator, are compared, showing that a detailed analysis of the projections of the support hull yields a tighter upper bound. The paper also elaborates on many of the results in [Chtcherba and Kapur, 2002d] providing with detailed proofs and demonstrates how these results strictly generalize our earlier results about the bivariate case.

In this paper, the focus is on the use of the generalized Dixon resultant formulation for computing resultants and projection operators. But the results apply to the Dixon multiplier matrices as well, since it is proved in [Chtcherba and

Kapur, 2002b] that given a generic unmixed polynomial system, if the Dixon formulation produces a Dixon matrix whose determinant is the resultant, then the corresponding Dixon multiplier matrix based on the construction in [Chtcherba and Kapur, 2002b] also has the resultant as their determinants. In case the Dixon matrix is such that the determinant of the maximal minor has an extraneous factor besides the resultant, the Dixon multiplier matrix does not have an extraneous factor of higher degree. (A Dixon multiplier matrix is a Sylvester-type resultant sparse matrix in which entries are either zeros or coefficients of terms in polynomials in the polynomial system, and it is constructed using the Dixon formulation; for more details, see [Chtcherba and Kapur, 2002b].)

The next section discusses preliminaries and background – the concept of a multivariate resultant of a polynomial system, the support of a polynomial and the degree of the resultant as determined by the BKK bound based on the mixed volume of the Newton polytopes of the supports of the polynomials in a polynomial system. Section 3 is a review of the generalized Dixon formulation including the Dixon polynomial and Dixon matrix. The section concludes with a discussion of how the Cauchy-Binet expansion of determinants can be used to show that the Dixon polynomial and its support are related to the support of the polynomials in the polynomial system. The size of the Dixon matrix of a polynomial system is determined by the size of the support of the associated Dixon polynomial. It is also shown how the support of the Dixon polynomial is affected when the support of the given polynomial system is translated. The rest of the paper is about generic unmixed polynomial systems whose support is cornered, i.e., situated at the origin (meaning that every polynomial includes a constant term).

Section 4 discusses the concept of a support hull and its interior. This concept turns out to be more useful for relating the size of the Dixon matrix of a given polynomial system to its support. It has been established in [Chtcherba and Kapur, 2002c] that the generic inclusion of a term whose exponent is support hull interior of the support of a given generic unmixed polynomial system does not change the size of the Dixon matrix of the modified polynomial system.

Section 5 reviews the results about generic unmixed bivariate polynomial systems. The concept of a corner-cut support is discussed; for generic unmixed polynomial systems, the support-hull of their support being corner-cut is both a necessary and sufficient condition for the Dixon-based resultant methods to compute resultants without any extraneous factors.

Section 6 generalizes the concepts and notations introduced to study bivariate polynomial systems to arbitrary dimension. By a combinatorial analysis of the deviation of a given support from that of an n-degree polynomial system, conditions are identified on a support for which the generalized Dixon formulation computes exact resultants (up to a sign). By considering projections of the support complement of the support of a given polynomial system with respect to the associated n-degree systems (whose support is the bounded box support of the support of the polynomial system) using a given variable order, a formula is derived for the size of the Dixon matrix in terms of the size of the Dixon matrix for the associated n-degree system and the size of various projections of the support complement.

Section 7 discusses conditions on the support of a polynomial system and their projections which lead to a Dixon matrix with the appropriate size such that its determinant is the resultant. The concept of a d-dimensional corner cut support is introduced which generalizes the notion of corner-cut support for the bivariate case discussed in [Chtcherba and Kapur, 2002e, Chionh, 2001, Zhang and Goldman, 2000]. It is shown that for a generic unmixed polynomial system with a corner-cut support, the Dixon formulation computes the resultant exactly. The requirement in the structural condition defining a corner-cut support can be relaxed while still preserving the property of the associated Dixon matrix that its determinant is the resultant; the concept of an *almost* corner-cut support (which have no analog in the bivariate case) is introduced. For a generic unmixed polynomial system with an almost corner-cut support as well, the Dixon method produces exact resultant. The notion of support-interior point within a support is generalized from the bivariate case to d variables; it is shown that a given unmixed polynomial system can be modified to generically include the term corresponding to a support-interior point in its support without affecting the size of the Dixon matrix.

2. Support and Degree of the Resultant

The resultant is defined for over-constrained polynomial system $\mathcal{F} = \{f_0, \ldots, f_d\}$ with

$$f_0 = \sum_{\alpha \in \mathcal{A}_0} c_{0,\alpha} \mathbf{x}^\alpha, \quad f_1 = \sum_{\alpha \in \mathcal{A}_1} c_{1,\alpha} \mathbf{x}^\alpha, \qquad \cdots, \qquad f_d = \sum_{\alpha \in \mathcal{A}_d} c_{d,\alpha} \mathbf{x}^\alpha,$$

where $\mathcal{A}_i \subset \mathbb{N}^d$, and a monomial $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$. In general, the structure of the resultant is dependent on the set of monomials appearing in the polynomial system. $\mathcal{A}_i$ is called the *support* of a polynomial $f_i \in \mathbb{Q}[\mathbf{c}][x_1, \ldots, x_d]$.

The (lattice) convex hull of the support of a polynomial f is called its Newton polytope, and will be denoted as $\mathcal{N}(f)$. One can relate the Newton polytopes of a polynomial system to the number of its roots, but first, we define a special function on supports, called the *mixed volume*.

Definition 2.1 ([Cox et al., 1998],[Gelfand et al., 1994]): *The* ***mixed volume function*** $\mu(\mathcal{Q}_1, \ldots, \mathcal{Q}_d)$*, where* $\mathcal{Q}_i$ *is a convex hull, is a unique function which is multi-linear with respect to Minkowski sum and scaling, and is defined to have the multi-linear property*

$$\mu(\mathcal{Q}_1, \ldots, a\mathcal{Q}_k + b\mathcal{Q}_k', \ldots, \mathcal{Q}_d) = \\ a\,\mu(\mathcal{Q}_1, \ldots, \mathcal{Q}_k, \ldots, \mathcal{Q}_d) + b\,\mu(\mathcal{Q}_1, \ldots, \mathcal{Q}_k', \ldots, \mathcal{Q}_d);$$

to ensure uniqueness, $\mu(\mathcal{Q},\ldots,\mathcal{Q}) = d!\mathsf{Vol}(\mathcal{Q})$*, where* $\mathsf{Vol}()$ *is the Euclidean volume measure.*

THEOREM 2.1 (BKK BOUND): *Given a polynomial system* $\{f_1,\ldots,f_d\}$ *in* d *variables* $\{x_1,\ldots,x_d\}$ *with the support* $\langle\mathcal{A}_0,\ldots,\mathcal{A}_d\rangle$*, the number of roots in* $(\mathbb{C}^*)^d$*, counting multiplicities, of the polynomial system is either infinite or*

$$\#\mathsf{Roots}(f_1,\ldots,f_d) \le \mu\left(\mathcal{A}_1,\ldots,\mathcal{A}_d\right);$$

the inequality becomes equality when the coefficient of polynomials in the system satisfy the genericity requirements.

Since we are interested in over-constrained polynomial systems, usually consisting of $d+1$ polynomials in d variables, the BKK bound tells us the degree of the resultant.

In the resultant, the degree of the coefficients of f_0 is equal to the number of common roots the rest of polynomials have. It is possible to choose any f_i and the resultant expression can be expressed by substituting in f_i, the common roots of the remaining polynomial system. This implies that in the resultant, the degree of the coefficients of f_i equals the number of roots of the remaining set of polynomials.

DEFINITION 2.2: *A polynomial system* $\mathcal{F} = \{f_0,\ldots,f_d\}$ *with the corresponding supports* $\mathcal{A}_0,\ldots,\mathcal{A}_d$*, is called* ***unmixed*** *if* $\mathcal{A}_0 = \mathcal{A}_1 = \cdots = \mathcal{A}_d$*, and* ***mixed*** *otherwise.*

In this article, we are primarily concerned with unmixed polynomial systems in which all polynomials have the same structure. Therefore, for notational convenience, we will drop the index of a supports $\mathcal{A} = \mathcal{A}_i$, and say that a polynomial system has the support $\mathcal{A}$, if every polynomial in it has the support $\mathcal{A}$.

For an unmixed polynomial system with a support $\mathcal{A}$, there is an easy formula for the degree of the resultant.

$$\deg_{f_i} \mathrm{Res} = d!\mathsf{Vol}(\mathcal{A}),$$

where $\deg_{f_i} \mathrm{Res}$ is the degree of the toric resultant in terms of coefficients of polynomial f_i. Knowing the degree of the resultant a priori will be useful for identifying cases for which a given method used to compute the resultant is exact, in the sense that the method does not produce a result with any extraneous information.

We first give a brief overview of the Dixon formulation, define the concepts of the Dixon polynomial and the Dixon matrix of a given polynomial system. Expressing the Dixon polynomial using the Cauchy-Binet expansion of determinants of a matrix is useful for illustrating the dependence of the construction on the support of a given polynomial system.

3. The Dixon Matrix

In [Dixon, 1908], Dixon generalized Bezout-Cayley's construction for computing the resultant of two univariate polynomials to the bivariate case. In [Kapur et al., 1994], Kapur, Saxena and Yang further generalized this construction to the general multivariate case; the concepts of a Dixon polynomial and a Dixon matrix were introduced as well. Below, the generalized multivariate Dixon formulation for simultaneously eliminating many variables from a polynomial system and computing its resultant are reviewed. More details can be found in [Kapur and Saxena, 1995].

In contrast to multiplier matrices such as a Sylvester matrix, Macaulay matrix and sparse matrices a la Sturmfels et al as well as Canny and Emiris, the Dixon matrix is dense since its entries are determinants of the coefficients of the polynomials in the original polynomial system. It has the advantage of being an order of magnitude smaller in comparison to a multiplier matrix, which makes the method efficient since the computation of the determinant of a matrix with symbolic entries is sensitive to its size. The Dixon matrix is constructed through the computation of the Dixon polynomial, which can be expressed in matrix form.

Let $\pi_i(\mathbf{x}^\alpha) = \overline{x}_1^{\alpha_1} \cdots \overline{x}_i^{\alpha_i} x_{i+1}^{\alpha_{i+1}} \cdots x_d^{\alpha_d}$, where $i \in \{0, \ldots, d\}$, and $\overline{x}_i$'s are new variables; $\pi_0(\mathbf{x}^\alpha) = \mathbf{x}^\alpha$. π_i is extended to polynomials in a natural way as:

$$\pi_i(f(x_1, \ldots, x_d)) = f(\overline{x}_1, \ldots, \overline{x}_i, x_{i+1}, \ldots, x_d),$$

obtained by substituting $\overline{x}_j$ for x_j in $f, 1 \le j \le i$.

Definition 3.1: *Given a polynomial system $\mathcal{F} = \{f_0, f_1, \ldots, f_d\}$, where for $\mathbf{c} = (c_{i,\alpha})$, $\mathcal{F} \subset \mathbb{Q}[\mathbf{c}][x_1, \ldots, x_d]$, define its* ***Dixon polynomial*** *as*

$$\theta(f_0, \ldots, f_d) = \prod_{i=1}^{d} \frac{1}{\overline{x}_i - x_i} \begin{vmatrix} \pi_0(f_0) & \pi_0(f_1) & \cdots & \pi_0(f_d) \\ \pi_1(f_0) & \pi_1(f_1) & \cdots & \pi_1(f_d) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_d(f_0) & \pi_d(f_1) & \cdots & \pi_d(f_d) \end{vmatrix}. \tag{1}$$

Hence $\theta(f_0, f_1, \ldots, f_d) \in \mathbb{Q}[\mathbf{c}][x_1, \ldots, x_d, \overline{x}_1, \ldots, \overline{x}_d]$, where $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_d$ are new variables.

The order in which the original variables in $\mathbf{x}$ are replaced by new variables in $\overline{\mathbf{x}}$ is significant in the sense that the Dixon polynomial computed using two different orderings may be different.

Definition 3.2: *The Dixon polynomial $\theta(f_0, \ldots, f_d)$ can be written in* ***bilinear form*** *as*

$$\theta(f_0, f_1, \ldots, f_d) = \overline{X}\Theta X^T,$$

where $\overline{X} = [\overline{\mathbf{x}}^{\beta_1}, \ldots, \overline{\mathbf{x}}^{\beta_k}]$ and $X = [\mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_l}]$ are row vectors. The $k \times l$ matrix Θ is called the ***Dixon matrix****.*

Since Θ is a resultant matrix (see [Kapur and Saxena, 1996] and [Buse et al., 2000]), the resultant is extracted from a *projection operator*, which is a determinant of some maximal minor of Θ.

Each entry in Θ is a polynomial in the coefficients of the original polynomials in $\mathcal{F}$; moreover, its degree in the coefficients of any given polynomial is at most 1. Therefore, the projection operator computed using the Dixon formulation can be of at most of degree $|X|$ in the coefficients of any single polynomial.

We are interested in identifying conditions when the resultant matrix Θ is **exact**, i.e., its determinant is exactly (up to a constant factor) the resultant. Also, when it is not, we are interested in predicting the extraneous factor in the determinant of Θ (at the very least, the degree of the extraneous factor). In the unmixed case,

$$|X| \geq d!\ \mathsf{Vol}(\mathcal{A}).$$

We are thus interested in analyzing the size and structure of the monomial set X; its size tells the number of columns in Θ and hence, whether or not, Θ is exact, which is the case when $|X| = d!\ \mathsf{Vol}(\mathcal{A})$.

We will relate the support $\mathcal{A}$ of a given unmixed polynomial system $\mathcal{F}$ to the support of its Dixon polynomial X and hence, the size of the Dixon matrix.

3.1. Relating Size of Dixon Matrix to Support of a Polynomial System

There is a different formula for the Dixon polynomial based on Cauchy-Binet expansion of the determinant of product of two non-square matrices.

Proposition 3.1 (Cauchy-Binet expansion): *Let $\mathcal{F} = \{f_0, f_1, \ldots, f_d\}$ be a polynomial system and let $\mathcal{A}$ be the support of $\mathcal{F}$. Then*

$$\theta(f_0, f_1, \ldots, f_d) = \sum_{\substack{\sigma \subset \mathcal{A} \\ |\sigma| = d+1}} \sigma(\mathbf{c})\ \sigma(\mathbf{x}) = \sum_{\substack{\sigma \subset \mathcal{A} \\ |\sigma| = d+1}} \theta_\sigma,$$

where $\theta_\sigma = \sigma(\mathbf{c})\ \sigma(\mathbf{x})$ and

$$\sigma(\mathbf{c}) = \begin{vmatrix} c_{0,\sigma_0} & c_{0,\sigma_1} & \cdots & c_{0,\sigma_d} \\ c_{1,\sigma_0} & c_{1,\sigma_1} & \cdots & c_{1,\sigma_d} \\ \vdots & \vdots & \ddots & \vdots \\ c_{d,\sigma_0} & c_{d,\sigma_1} & \cdots & c_{d,\sigma_d} \end{vmatrix},$$

$$\sigma(\mathbf{x}) = \prod_{i=1}^{d} \frac{1}{\overline{x}_i - x_i} \begin{vmatrix} \pi_0(\mathbf{x}^{\alpha_{\sigma_0}}) & \pi_0(\mathbf{x}^{\alpha_{\sigma_1}}) & \cdots & \pi_0(\mathbf{x}^{\alpha_{\sigma_d}}) \\ \pi_1(\mathbf{x}^{\alpha_{\sigma_0}}) & \pi_1(\mathbf{x}^{\alpha_{\sigma_1}}) & \cdots & \pi_1(\mathbf{x}^{\alpha_{\sigma_d}}) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_d(\mathbf{x}^{\alpha_{\sigma_0}}) & \pi_d(\mathbf{x}^{\alpha_{\sigma_1}}) & \cdots & \pi_d(\mathbf{x}^{\alpha_{\sigma_d}}) \end{vmatrix}.$$

Proof: See [Chtcherba and Kapur, 2002b] for a detailed proof. □

The above identity shows that if generic coefficients are assumed in the polynomial system, then the support of the Dixon polynomial depends entirely on the support of the polynomial system, as $\sigma(\mathbf{c})$ would not vanish or cancel each other. To emphasize the dependence of θ on $\mathcal{A}$, the above identity can also be written as $\theta_{\mathcal{A}} = \sum_{\sigma \subset \mathcal{A}} \theta_\sigma$.

Define the support of the Dixon polynomial as:

$$\Delta_{\mathcal{A}} = \{\, \alpha \mid \mathbf{x}^\alpha \in \theta(f_0, \dots, f_d) \,\}.^{\dagger}$$

And,

$$\Delta_{\mathcal{A}} = \bigcup_{\substack{\sigma \subset \mathcal{A} \\ |\sigma| = d+1}} \Delta_\sigma, \qquad \text{where} \quad \Delta_\sigma = \{\, \alpha \mid \mathbf{x}^\alpha \in \theta_\sigma \,\}.$$

The following proposition shows that the translation of the support of polynomials in an unmixed system has no effect on the size of the support of the Dixon polynomial (and hence the size of the Dixon matrix).

Proposition 3.2: *Given an unmixed polynomial system with support $\mathcal{A}$, let $q = (q_1, \dots, q_d)$, where $q_i = \min_{\alpha \in \mathcal{A}} \alpha_i$, then*

$$\Delta_{\mathcal{A}} = (q_1, 2\,q_2, \dots, dq_d) + \Delta_{\mathcal{A}-q},^{\ddagger}$$

that is $\Delta_{\mathcal{A}}$ is a "shift" of the support of the Dixon polynomial of the support situated at the origin.

Proof: Since $\mathcal{A}$ is the support of polynomials $\{f_0, f_1, \dots, f_d\}$, it follows that

$$f_0 = \mathbf{x}^q g_0, \quad f_1 = \mathbf{x}^q g_1, \quad \cdots, \quad f_2 = \mathbf{x}^q g_d,$$

where $\mathcal{A} - q$ is the support of $\{g_0, g_1, \dots, g_2\}$. Therefore

$$\theta(f_0, f_1, \dots, f_d) = x_1 x_2^{2\,q_2} \cdots x_d^{dq_d}\, \overline{x}_1^{d\,q_1}\, \overline{x}_1^{(d-1)q_2} \cdots \overline{x}_d\, \theta(g_0, g_1, g_2),$$

by factoring monomials from the rows of the matrix in the expression for the Dixon polynomial as given in (1). Hence the statement. $\square$

Henceforth, it will be assumed without any loss of generality that in the unmixed case, $\mathcal{A}$ is situated at the origin, that is, $\min_{\alpha \in \mathcal{A}} \alpha_i = 0$ for $i = 1, \dots, d$

Below, the results of [Zhang and Goldman, 2000],[Chionh, 2001] and [Chtcherba and Kapur, 2002e], which identify supports under which Dixon matrix is of exact size for the bivariate case, are generalized. These results have implications not only for computing resultants using the Dixon matrix, which is the focus of this report, but the results also apply to the use of Dixon multiplier matrices developed in [Chtcherba and Kapur, 2002b], for computing resultants.

†By abuse of notation, for some polynomial f, by $\mathbf{x}^\alpha \in f$ we mean that $\mathbf{x}^\alpha$ appears in (the simplified form of) f with a non-zero coefficient, i.e. α is in the support of f.

‡"$-$" is the regular vector subtraction.

4. Support Hull

In [Chtcherba and Kapur, 2002c], the concept of a support hull is introduced. It is shown that just like the degree of the resultant depends on the convex hull of the supports, the size of the Dixon matrix depends on the support hull of a support. These notions and relationships are reviewed here; more information can be found in [Chtcherba and Kapur, 2002c].

Geometrically, a hull is a closed set defined by a property specifying when a given point belongs to the hull. A comparison operator among points is defined, which is then used to characterize the support hull of a given support.

Given two points on a line one can say that one point is before the other in some direction. Going to higher dimensions, such relationship between the points can be extended rectilinearly as follows.

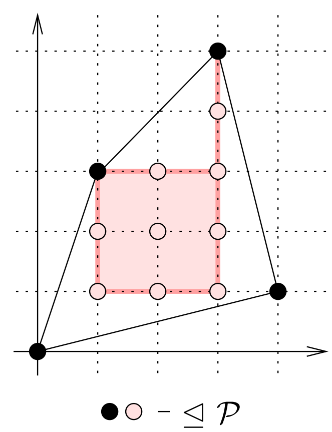

Figure 1: Support Hull

Definition 4.1: *Given $k \in \mathbb{Z}_2^d$ and points $p, q \in \mathbb{N}^d$, define*

$$p \underset{k}{<} q \quad if \quad \begin{cases} p_j < q_j & if\ k_j = 1, \\ p_j \geq q_j & if\ k_j = 0. \end{cases}$$

Any $k \in \mathbb{Z}_2^d$ is called an octant; if $d = 2$, it is called a quadrant. Note that from above, p_j is strictly smaller that q_j; from below, p_j is equal or greater that q_j. When equality is also allowed from above, the relation is denoted by $p \underset{k}{\leq} q$. For a fixed k, this relation is transitive, but it is not a total order.

Definition 4.2: *Given a support $\mathcal{P} \subset \mathbb{N}^d$, define its* ***support hull***

$$\texttt{SupportHull}(\mathcal{P}) = \{\, p \mid \forall\, k \in \mathbb{Z}_2^d,\ \exists\, q \in \mathcal{P},\ such\ that\ p \underset{k}{\leq} q \,\}.$$

For short, $p \trianglelefteq \mathcal{P}$ if $p \in$ `SupportHull`$(\mathcal{P})$.

The support hull of a given support is thus a minimal object rectilinearly connecting all points in a support. In contrast to the convex hull of a support, the support hull is not a connected set. Figure 1 shows the support (filled points) and its support hull (all points).

Definition 4.3: *A point $p \in \mathbb{N}^d$ is called* ***support hull interior*** *w.r.t. a support $\mathcal{A}$ if for all octants $k \in \mathbb{Z}_2^d$, there exists $q \in \mathcal{A}$, where $p \neq q$, such that $p \underset{k}{\leq} q$, i.e. every octant of p contains point from $\mathcal{A}$.*

From the definition, it can be easily be seen that the support hull interior points are convex hull interior. In general, two support hulls $\mathcal{P}$ and $\mathcal{Q}$ are equivalent if and only if for all p, $p \trianglelefteq \mathcal{P}$ implies $p \trianglelefteq \mathcal{Q}$.

Theorem 4.1: *[Chtcherba and Kapur, 2002c] Given two* unmixed *polynomial systems with cornered supports* $\mathcal{P}$ *and* $\mathcal{Q}$ *such that*

$$\texttt{SupportHull}(\mathcal{P}) = \texttt{SupportHull}(\mathcal{Q}), \quad \textit{then} \quad \Delta_{\mathcal{P}} = \Delta_{\mathcal{Q}},$$

i.e., the polynomial systems have Dixon matrices of the same size.

The following corollary is an immediate consequence of the above theorem.

Corollary 4.1: *The size of the Dixon matrix of a generic unmixed polynomial system is invariant under the generic presence of a monomial in the polynomial system whose exponent is support interior w.r.t. the support of the original polynomial system.*

Such dependence of the construction and size of the Dixon matrix on the support of a polynomial system suggests a way to identify polynomial systems for which the Dixon construction will result in projection operators with extraneous factors.

If the support of a generic unmixed polynomial system contains a point which is in the interior of the convex hull of the support, but is not support hull interior, the Dixon construction will result in a projection operator with an extraneous factor. Unfortunately, the converse does not hold since there are examples of generic unmixed systems with no such points for which the Dixon construction produces projection operators with extraneous factors. In subsequent sections, we will give a more precise description of supports which are exact under the Dixon construction. We first review known results for the bivariate case. We then generalize the concepts and results to the general d variable case.

5. Bivariate case: Corner Cut Systems

[Zhang and Goldman, 2000] has identified a necessary condition on polynomial system support under which it is possible to construct exact Sylvester-like resultant matrices. [Chionh, 2001] has shown that under this condition, the Dixon matrix is also exact. In [Chtcherba and Kapur, 2002e], we have generalized these results by establishing that this condition on supports is necessary and sufficient for both Sylvester-like matrices as well as Dixon matrices to produce resultants exactly. In addition, we introduced the concept of a support-interior point in a support with the property that the generic inclusion of a term whose exponent is a support-interior point, in a generic unmixed bivariate polynomial system does not change the size of the Dixon matrix. First, we review these concepts and results for the bivariate case and later, we show how they can be generalized for an arbitrary number of variables.

Given a cornered support $\mathcal{A}$, such that $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\}$, let $b_j = \max_{i=1}^{n} \alpha_{i,j}$, for $j = 1, \ldots, d$.

Definition 5.1: *Given a support $\mathcal{A}$, define the **box support** $\mathcal{B}$ of $\mathcal{A}$ as:*

$$\mathcal{B}_i = \{\ p = (p_1, \dots, p_d) \mid 0 \le p_j \le b_j\ \},$$

for all $i = \{0, \dots, d\}$.

An unmixed polynomial system with a box support $\mathcal{B}$ is called n-degree. It has been established in [Kapur and Saxena, 1996, Saxena, 1997] that for such systems Dixon matrix is exact.

For the bivariate case, consider the following set:

Definition 5.2: *Given a generic unmixed polynomial system with the support $\mathcal{A}$, let, for $k \in \mathbb{Z}_2^2$,*

$$S^k = \{\ s \mid s \in \mathcal{B} \ \text{ and for all } \alpha \in \mathcal{A},\ s \not\le_k \alpha\ \} \quad \text{and} \quad S = \bigcup_{k \in \mathbb{Z}_2^2} S^k.$$

As a consequence of Definitions 4.2 and 5.2, it is easy to see that

$$S = \mathcal{B} - \texttt{SupportHull}(\mathcal{A}).$$

See Figure 2, where the hollow points belong to S and the crossed points are in the support hull interior of $\mathcal{A}$.

Definition 5.3: *A bivariate polynomial system support $\mathcal{A}$ is called **corner-cut** if for each $k \in \mathbb{Z}_2^2$, the corresponding set S^k is* rectangular*.*

Theorem 5.1: *[Chtcherba and Kapur, 2002e] A generic bivariate unmixed polynomial system is Dixon exact if and only if its support is corner cut.*

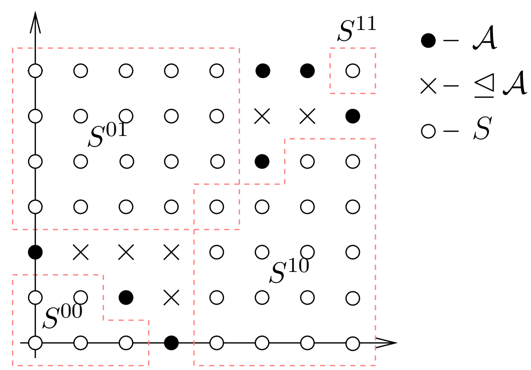

Figure 2: Support Hull Complement

In the later sections, we generalize the above theorem for the general case. It will be clear that a straight forward generalization is not appropriate. The main reason for this is the role of variable order played in the Dixon construction; this role is not evident in the bivariate case.

6. Generalizing Corner-Cut Supports

The complexity of the resultant computation using the Dixon formulation is governed by the size of the Dixon matrices generated for a given polynomial system. An upper bound on the size of the Dixon matrix for a generic unmixed polynomial system was proved in [Kapur and Saxena, 1996] to be the Minkowski sum of successive projections of the supports of the polynomials in the polynomial system. Depending on a variable order used in the construction (definition 3.1), different Dixon polynomials and hence, different Dixon matrices can be obtained, and more importantly, the size of the resulting Dixon matrices might not be the same. Dixon polynomials of smaller degrees will result in the smaller Dixon (as well as Dixon multiplier) matrices and hence, extraneous factors of smaller degrees in the projection operators.

In this section, a tighter bound on the size of the Dixon matrix is established by analyzing how the support of a given generic unmixed polynomial system differs from the support of the associated n-degree polynomial system.

6.1. n-Degree Systems and their Box Supports

It has been proved in [Kapur and Saxena, 1996] and [Saxena, 1997] that Dixon-based matrices are exact for generic polynomial systems with an n-degree support.

PROPOSITION 6.1: *The support of the Dixon polynomial of an n-degree system with a box support $\mathcal{B}$,*

$$\Delta_{\mathcal{B}} = \{\ p = (p_1, \ldots, p_d) \mid 0 \le p_i < i\ b_i\ \} \quad \textit{and} \quad |\Delta_{\mathcal{B}}| = d!\ \mathsf{Vol}(\mathcal{B}) = d! \prod_{i=1}^{d} b_i.$$

As in the case of bivariate systems, we relate the difference between the support $\Delta_{\mathcal{B}}$ of the Dixon polynomial of the system with the box support $\mathcal{B}$ to the support $\Delta_{\mathcal{A}}$ of the Dixon polynomial of the system with the support $\mathcal{A}$ to the difference between the box support $\mathcal{B}$ and the support $\mathcal{A}$. However, unlike the bivariate case, this difference will have to be analyzed by investigating projections of $\mathcal{A}$ along different coordinates. Unlike the bivariate case, it is not sufficient to analyze the projection using a single variable order. Such an analysis will allow us to establish a tight upper bound on the size of $\Delta_{\mathcal{A}}$ and hence, the size of the associated Dixon matrix.

6.2. Earlier Known Bounds [Kapur and Saxena, 1996]

In contrast to the bivariate case, the difference between $\mathcal{B}$ and $\mathcal{A}$ has to be described more precisely, taking into account the effect of the variable order on the construction of Dixon matrices.

It was proved in [Kapur and Saxena, 1996] that $\Delta_{\mathcal{A}}$ is contained in the convex hull of the Minkowski sum of the projections of $\mathcal{A}$.

THEOREM 6.1: *[Kapur and Saxena, 1996] Given a generic unmixed polynomial system with the support $\mathcal{A}$,*

$$\Delta_{\mathcal{A}} \subseteq \texttt{ConvexHull}\left(\sum_{i=0}^{d} \pi_i(\mathcal{A})\right)\ \cap\ \mathbb{Z}^d,$$

where $\pi_i(\mathcal{A}) = \{\,\pi_i(\alpha)\,|\,\alpha \in \mathcal{A}\,\}$ *and* $\pi_i(\alpha) = (0,\ldots,0,\alpha_{i+1},\ldots,\alpha_d)$ *for* $i \in \{0,\ldots,d\}$.

One can easily see that in the expression for the Dixon polynomial (definition 3.1), the support of the determinant of the matrix is contained in the Minkowski sum of projection, and division by $\overline{x}_i - x_i$ will not introduce any new points outside the convex hull of the sum. Consequently, since in general for some support $\mathcal{P}$, $|\mathtt{ConvexHull}(\mathcal{P}) \cap \mathbb{Z}^d| = O(\mathsf{Vol}(\mathcal{P}))$,

$$|\Delta_{\mathcal{A}}| = O\left(\mathsf{Vol}\left(\sum_{i=0}^{d}\pi_i(\mathcal{A})\right)\right).$$

6.3. Tighter Bounds using Support Hull

In the above formula, the "convex hull" of a given support can be replaced by its "support hull" without violating the theorem and the main result; this is due to the property that the Dixon matrix construction depends on the support hull of a support, [Chtcherba and Kapur, 2002c]. Hence,

$$|\Delta_{\mathcal{A}}| \leq \#\mathsf{Points}\left(\mathsf{SupportHull}\sum_{i=0}^{d}\pi_i(\mathcal{A})\right).$$

Even though a support hull is a "smaller" object than a convex hull (since it is contained in the convex hull), this does imply that a bound based on the support hull is always better than the corresponding bound based on the convex hull. Figure 3 is an example for which an upper bound based on the support hull is worse than the volume of the convex hull. (The shaded area in the figure corresponds to the support hull). The volume of the sum of projections (the outline of the figure) is 31.5, where as the number of points in the support hull of the sum of the projections (in the shaded area in the figure) is 38. (For this example, the size of the Dixon polynomial $|\Delta_{\mathcal{A}}| = 23$).

In general, however, a bound based on the support hull is usually better. Consider the case when 2D support contains only 3 points $\{(0,0),(9,0),(0,9)\}$. The volume of the support projections is 150 whereas the number of points in the support hull of projections is 109.

This examples suggest a need for further analysis to develop a tighter upper bound. Particularly, $\Delta_{\mathcal{A}}$ does not include "upper" boundary points of the support hull (hollow points in Figure 3). Because of the division in the construction of the Dixon polynomial, the degrees of monomials are reduced. When this is considered, an exact bound for the bivariate case can be obtained; see [Chtcherba and Kapur, 2002e] for a complete analysis.

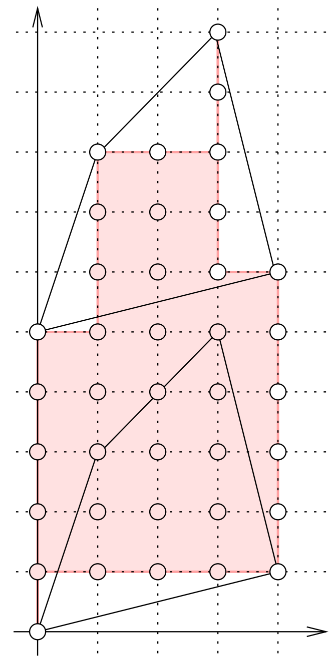

Figure 3: Support Hull of sum of projections

THEOREM 6.2: *[Chtcherba and Kapur, 2002c] Given an unmixed generic polynomial system with support $\mathcal{A}$, the size of Dixon matrix is bounded by the number of points in the support hull of the sum of projections minus "upper" boundary points.*

See [Chtcherba and Kapur, 2002c] for more a formal statement and a detailed proof.

Even though there is now an exact description of the points in the support of the Dixon polynomial, it is difficult to estimate the sizes of the Dixon polynomial and the Dixon matrix. An objective is to identify a direct relationship between the size of the Dixon matrix and the supports of the input polynomial system. Moreover, as has been observed experimentally as well as stated above, the size of the Dixon matrix depends on the variable order used to construct the Dixon polynomial. Therefore, we analyze the sum of projections under different variable orders to arrive at an upper bound on the size of the Dixon matrix; since this bound is tighter, in some cases, an exact bound is obtained.

6.4. Bound using Support Projections

In general, the Dixon matrix size is sensitive to the variable order used in the construction. Thus, projections along different direction need to be considered more carefully; instead of approximating the size of the Dixon matrix by the sum of projections, it is possible to get a much tighter bound that can be shown to be exact for many non-trivial cases of supports. In fact, the results below are the first to show the dependence of the Dixon matrix construction on different variable orders.

We define the following projection operations on an arbitrary support $\mathcal{A}$:

$$\mathcal{A}_{(l_1,\ldots,l_i)} = \Big\{\, \alpha' = (\alpha'_1,\ldots,\alpha'_d) \,\Big|\, \alpha'_j = \alpha_j \text{ if } j \in \{l_1,\ldots,l_i\}, \\ \text{and } \alpha'_j = 0 \text{ otherwise, for } \alpha \in \mathcal{A} \,\Big\}.$$

For example, if $d = 4$, then $\mathcal{A}_{(1,4)} = \{(\alpha_1, 0, 0, \alpha_4) \mid \text{for all } \alpha \in \mathcal{A}\}$. This allows us to model various variable orders. $\mathcal{A}_{(1,4)}$ notes that first and fourth variables occur before second and third. Hence $\pi_i(\mathcal{A}) = \mathcal{A}_{(l_{i+1},\ldots,l_d)}$ and $\mathcal{A}_{(1\ldots d)} = \mathcal{A}$. Below, we introduce the following notation:

$$\mathcal{A}_{(l_1\ldots l_i,*)} = \Big\{\, \alpha' = (\alpha'_1,\ldots,\alpha'_d) \,\Big|\, \alpha'_j = \alpha_j \text{ if } j \in \{l_1,\ldots,l_i\}, \\ \text{and } 0 \le \alpha'_j \le b_j \text{ otherwise, for } \alpha \in \mathcal{A} \,\Big\},$$

that is, the coordinates which are not specified, can assume any value within the range of the bounding box.

For convenience, we will also assume below that a given support $\mathcal{A}$ is equal to its support hull.

We will look how different $\mathcal{A}$ is from $\mathcal{B}$ for each projection. Consider the complement of $\mathcal{A}$ in $\mathcal{B}$. This difference between $\mathcal{B}$ and $\mathcal{A}$ is analyzed by considering their successive projections along coordinates. Let

$$\mathcal{C}_{(l_1\ldots l_i)} = \mathcal{B}_{(l_1\ldots l_i)} - \mathcal{A}_{(l_1\ldots l_i)}.$$

Note in the bivariate case, $\mathcal{C}_{(1,2)}$ is exactly the support complement. It should be noted that $\mathcal{C}_{(i)} = \oslash$ for any $i \in \{1,\ldots,d\}$.

See, for example, Figure 4, where a 3 dimensional support $\mathcal{A}$ and two coordinate projection $\mathcal{A}_{(1,2)}$ and $\mathcal{A}_{(1,3)}$ are shown. By selecting the coordinates, a variable order used in the Dixon construction can be modeled. If $\mathcal{A}_{(1,2)}$ is considered, then the variable order is $\langle x, y, z\rangle$; in case $\mathcal{A}_{(1,3)}$ is considered, the variable order is $\langle x, z, y\rangle$. As is evident in the figure, these projections are sensitive to the coordinate order.

$\mathcal{C}_{(l_1,\ldots,l_i)}$ is the difference of the bounding box $\mathcal{B}_{(l_1,\ldots,l_i)}$ and $\mathcal{A}_{(l_1,\ldots,l_i)}$ in $l_1,\ldots,l_i$-dimensions. For instance, $\mathcal{C}_{(1,2)}$ in Figure 4 has two points $(3,3)$ and $(3,4)$ corresponding to the variable order $\langle x, y, z\rangle$; in case the variable order $\langle x, z, y\rangle$ is used, then $\mathcal{C}_{(1,3)}$ has only one point, $(3,3)$. Therefore, the choice of a variable order is important in analyzing the complement of $\mathcal{A}$ with respect to $\mathcal{B}$.

Just in the bivariate case, we define the support complement to define an upper bound on the size of the Dixon polynomial.

Definition 6.1: *Given a support $\mathcal{A}$, and a list of coordinates $(l_1,\ldots,l_i)$, define its **support complement***

$$S_i = \mathcal{C}_{(l_1\ldots l_i)} - \mathcal{C}_{(l_1\ldots l_{i-1},*)},$$

and also define

$$S_{(i,*)} = \mathcal{C}_{(l_1\ldots l_i,*)} - \mathcal{C}_{(l_1\ldots l_{i-1},*)}.$$

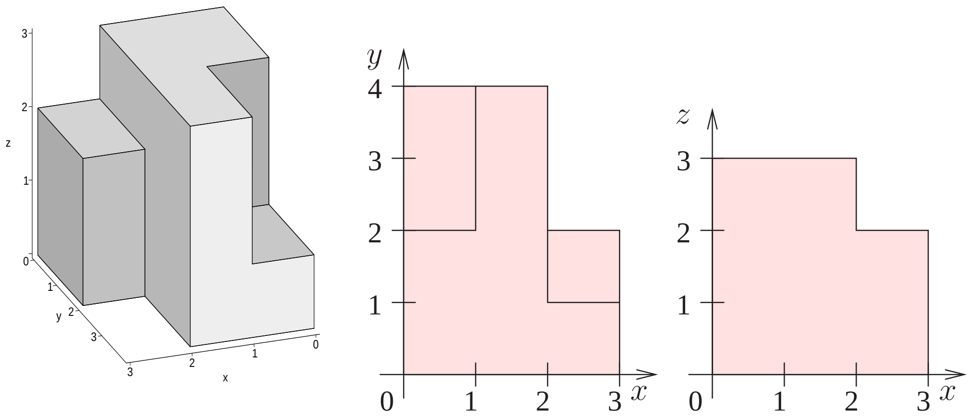

Figure 4: A support $\mathcal{A}$ and its $\{x, y\}$ and $\{x, z\}$ projections.

Clearly $S_i \subseteq S_{(i,*)}$, and set $S_{(i,*)}$ includes all the points from the bounding box, whose first i coordinates, match first i coordinates of any point from set S_i.

Note that the definition of S_i is always with respect to some $l_1, \ldots, l_i$, which is usually implied. Also note, that in the bivariate case, $\mathcal{C}_{(1,2)} = S_2 = S$ and hence notation is consistent with section 5. We will call set (S_i) for $i = 1, \ldots, d$, the i^{th} **support complement**. Informally, S_i attempts to include only those points of $\mathcal{C}_{(l_1 \ldots l_i)}$ which are not already included in $S_{(j,*)}, j < i$.

In the example of Figure 4, to compute S_3, first $\mathcal{C}_{(1,2,3)}$ has to be computed, which is composed of 14 points which can be identified from the Figure. $S_3 = \mathcal{C}_{(1,2,3)} - \mathcal{C}_{(1,2,*)}$ contains 6 points if the variable order $\langle x, y, z\rangle$ is used, and these points are $(3, 0, 3)$, $(3, 1, 3)$ and $(0, 3, 2), (0, 4, 2), (0, 3, 3), (0, 4, 3)$. Note $S_{(2,*)} = \mathcal{C}_{(1,2,*)}$ is composed of 8 points which are $(3, 3, *), (3, 4, *)$.

Below, we first list a few properties of $\mathcal{C}_{(l_1,\ldots,l_i)}$, $\mathcal{C}_{(l_1,\ldots,l_i,*)}$, S_i and $S_{(i,*)}$.

PROPOSITION 6.2: *Sets S_i, $S_{(i,*)}$ and $\mathcal{C}_{(l_1,\ldots,l_i)}$, as defined above, satisfy the following properties,*

1. *In all cases, $|S_1| = 0$, because $\mathcal{C}_{(l_i)} = \oslash$, as $\mathcal{B}_{(l_i)} = \mathcal{A}_{(l_i)}$ for any $l_i \in \{1, \ldots, d\}$.*
2. *For $j < i$, $\mathcal{C}_{(l_1,\ldots,l_j,*)} \cap \mathcal{B} \subseteq \mathcal{C}_{(l_1,\ldots,l_i)}$.*
3. *$\mathcal{C}_{(l_1,\ldots,l_i,*)} \subseteq \mathcal{C}_{(l_1,\ldots,l_{i-1},*)}$ and $S_{(i,*)} \subseteq \mathcal{C}_{(l_1,\ldots,l_{i-1},*)}$.*
4. *$S_{(i,*)} \cap S_{(j,*)} = \oslash \quad$ for $\quad i \neq j$.*

The following proposition states that the support $\mathcal{A}$ is cut out from $\mathcal{B}$ by all S_i's.

PROPOSITION 6.3: *Given an unmixed support hull $\mathcal{A}$ with its bounding box $\mathcal{B}$,* *sets $S_{(i,*)}$ as defined above for $i = 1, \ldots, d$, then*

$$\mathcal{A} = \mathcal{B} - \bigcup_{i=1}^{d} S_{(i,*)}.$$

Proof: Clearly $S_{(i,*)} \subset \mathcal{B}$ and $S_{(i,*)} \cap \mathcal{A} = \oslash$. We only need to show that if $p \notin \mathcal{A}$ then $p \in S_{(i,*)}$ for some $i \in \{1, \ldots, d\}$.

Since $p \notin \mathcal{A}$ then $p \in \mathcal{C}_{l_1,\ldots,l_d}$. Then since

$$S_d = \mathcal{C}_{(l_1,\ldots,l_d)} - \mathcal{C}_{(l_1,\ldots,d-1,*)},$$

then $p \in S_d$ or $p \in \mathcal{C}_{(l_1,\ldots,l_{d-1},*)}$. In the former case we have same relationship where $p \in S_{d-1}$ or $p \in \mathcal{C}_{(l_1,\ldots,l_{d-2},*)}$. Since $S_{(1,*)} = \mathcal{C}_{(l_1,*)}$, p must belong to some $S_{(i,*)}$. □

The sets $S_{(i,*)}$ are structured in such a way to model the projections. All the points in $S_{(i,*)}$ cannot be projected onto $S_{(i-1,*)}$. So, for each projection, only those points outside support hull $\mathcal{A}$ which cannot be projected onto lower dimension need to be considered. Further, as in the bivariate case, these points are partitioned into disjoint sets which are located in the separate corners of $\mathcal{B}$.

6.4.1. Splitting Support Complement

Definition 6.2: *For every $k \in \mathbb{Z}_2^i$, define S_i^k as:*

$$S_i^k = \left\{ p \mid p \in S_i \quad and \quad \nexists\, \alpha \in \mathcal{A}_{(l_1 \ldots l_i)} \quad s.t. \quad p \underset{k}{\leq} \alpha \right\}. \tag{2}$$

In Figure 5, for $k = (0, 1, 1)$, $S_3^k = \{(0, 0, 3), (0, 1, 3)\}$ and for $k = (1, 1, 1)$, S_3^k is composed of 4 points, $\{(3, 2, 2), (3, 2, 3), (3, 3, 2), (3, 3, 3)\}$. For all other $k \in \mathbb{Z}^3$, $S_3^k = \oslash$ for the example. Since $S_2 = S_2^{(0,1)} = \{(0, 2), (0, 3)\}$, we have only one part for S_2 when $k = (0, 1)$. Hence, $S_{(2,*)}$ is composed of 8 points which are $\{(0, 2, 0), (0, 3, 0), (0, 2, 1), (0, 3, 1), (0, 2, 2), (0, 3, 2), (0, 2, 3), (0, 3, 3)\}$ (hollow points in Figure 5).

Proposition 6.4: *For any $i \in \{1, \ldots, d\}$,*

$$S_i = \bigcup_{k \in \mathbb{Z}_2^i} S_i^k \quad but \quad |S_i| \leq \sum_{k \in \mathbb{Z}_2^i} |S_i^k|,$$

that is, S_i^k's are not necessarily disjoint for different k's.

Proof: Let $p \in S_i$, then $p \in \mathcal{B}_{(l_1 \ldots l_i)} - \mathcal{A}_{(l_1 \ldots l_i)}$, and hence $p \notin \mathcal{A}_{(l_1 \ldots l_i)}$. Therefore, there exists $k \in \mathbb{Z}_2^i$ such that $\forall\, \alpha \in \mathcal{A}_{1 \ldots i}$, $p \not\leq_k \alpha$, and hence by definition 6.2, $p \in S_i^k$. $\square$

The case when S_i^k are not disjoint can be easily seen in the bivariate case. For example, Figure 2 shows such a case when $S_2^{(1,0)}$ has an nonempty intersection with $S^{(0,1)}$.

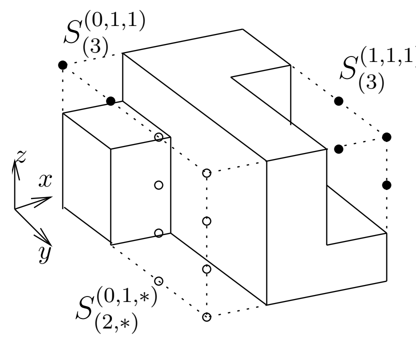

Figure 5: Sets $S^k_{(i,*)}$.

6.4.2. Relating the size of support complement to size of Dixon matrix

Just as for $\mathcal{A}$ for which the support complement $S_{(i,*)}$ is a part which is "missing" from $\mathcal{B}$, there is a corresponding part for $\Delta_{\mathcal{A}}$ which is "missing" from $\Delta_{\mathcal{B}}$. This is similar to the analysis done for the bivariate case in [Chtcherba and Kapur, 2002e]. We will relate $S_{(i,*)}$ with $\Delta_{\mathcal{B}} - \Delta_{\mathcal{A}}$.

Definition 6.3: *For any $k \in \mathbb{Z}_2^d$ and $r^k = (r_1^k, \ldots, r_d^k) \in \mathbb{N}^d$, let*

$$T_i^k = r^k + S_i^k \quad \textit{and} \quad T_i = \bigcup_{k \in \mathbb{Z}_2^d} T_i^k \quad \textit{where} \quad r_j^k = \begin{cases} (j-1)\, b_j - 1 & \textit{if } k_j = 1 \\ 0 & \textit{if } k_j = 0 \end{cases}.$$

Using the same notation, define $T_{(i,*)}^k = r^k + S_{(i,*)}^k$ and $T_{(i,*)} = \bigcup T_{(i,*)}^k$. The crucial point here is that lattice points in $T_{(i,*)}$ are not in $\Delta_{\mathcal{A}}$.

Proposition 6.5: *Given a generic unmixed polynomial system with the support $\mathcal{A}$, let $\mathcal{B}$ be its bounding box support. Then for any $i \in \{1, \ldots, d\}$,*

$$T_i^* \subseteq \Delta_{\mathcal{B}} - \Delta_{\mathcal{A}}.$$

Proof: See [Chtcherba and Kapur, 2002c]. $\square$

We thus obtain a much smaller set than $\Delta_{\mathcal{B}}$ in which the support of the Dixon polynomial $\Delta_{\mathcal{A}}$ is included.

Theorem 6.3:

$$\Delta_{\mathcal{A}} \subseteq \Delta_{\mathcal{B}} - \bigcup_{i=1}^{d} T_i^*.$$

Proof: Since $\Delta_{\mathcal{A}} \subseteq \Delta_{\mathcal{B}}$ and if $p \in \Delta_{\mathcal{A}}$, then $p \notin T_i$ by Proposition 6.5. $\square$

As shown in [Chtcherba and Kapur, 2002e], the above relation becomes equality for $d = 2$; but, in general, the relation is that of a subset. Later we identify classes of support for which the above relation becomes equality. An example in Figure 4 is a case where $\Delta_{\mathcal{A}}$ is strictly a subset. This is evident from the analysis of the Minkowski sum of projections since T_i's do not properly account for that.

To estimate an upper bound on the size of the support of the Dixon polynomial and hence the size of the Dixon matrix, a number of properties of T_i's are needed.

Proposition 6.6: *For $i, j \in \{0, \ldots, d\}$ s.t. $i \neq j$ and $k, l \in \mathbb{Z}_2^i$ such that $k \neq l$,*

(i) $T_i = \bigcup_{k \in \mathbb{Z}_2^i} T_i^k$,

(ii) $\left|T_i^k\right| = \left|S_i^k\right|$,

(iii) $T_i^k \cap T_i^l = \emptyset$,

(iv) $T_{(i,*)} \cap T_{(j,*)} = \emptyset$,

Proof: Statements (i) and (ii) follow from Definition 6.3.
Statement (iii): The proof is done by contradiction. Suppose there exists $p \in T_i^k \cap T_i^l$. By Definition 6.3,

$$s + r^k = p = q + r^l \quad \text{for} \quad s \in S_i^k \quad \text{and} \quad q \in S_i^l.$$

Since $k \neq l$, let j be such smallest integer such that $k_j \neq l_j$, w.l.o.g. let $k_j = 0$ and $l_j = 1$, then

$$s_j = p_j = q_j + \underbrace{(j-1)b_j - 1}_{r^l},$$

where b_j is maximum value of j^{th} coordinate for all points in $\mathcal{A}$. Since $s \in S_i^k$ there exists $\alpha \in \mathcal{A}$ such that $s_j < \alpha_j \leq b_j$, therefore the above equality might hold only for $j = 1$ or $j = 2$.

If k and l disagree only on one coordinate then by Definition $0 \leq s_j < \alpha_j < q_j \leq b_j$, making the equality impossible. If k and l disagree on first two coordinates, then assuming that $k_1 = 0$, $l_1 = 1$, would imply that

$$s_1 = q_1 - 1 \quad \text{and} \quad \begin{cases} s_2 = q_2 + b_2 - 1 & \text{if } k_2 = 0 \\ s_2 + b_2 - 1 = q_2 & \text{if } k_2 = 1 \end{cases}$$

If $k_2 = 0$ then $q_2 = 0$ as $s_2 < b_2$, but then since $l_2 = 1$, it cannot be the case that $q \in S_i^l$; on the other hand if $k_2 = 1$, then s has the same problem. Therefore, there is no such p in $T_i^k \cap T_i^l$ and hence $T_i^k \cap T_i^l = \emptyset$.
Statement (iv): Again, the proof is by contradiction. Suppose $T_{(i,*)} \cap T_{(j,*)} \neq \emptyset$ and $p \in T_{(i,*)} \cap T_{(j,*)}$. Then, $p \in T_{(i,*)}^k$ and $p \in T_{(j,*)}^l$ for some $k, l \in \mathbb{Z}_2^d$. By definition 6.3,

$$p = r^k + s^k \quad \text{for some} \quad s^k \in S_i^k \qquad \text{and} \qquad p = r^l + s^l \quad \text{for some} \quad s^l \in S_j^l.$$

Note that $k \neq l$ since, as noted earlier $S_i \cap S_j = \emptyset$.

Assume w.l.o.g. that $k_0 \neq l_0$ and $k_0 = 0$ where $l_0 = 1$. Then there exists $\alpha \in \mathcal{A}$ such that

$$0 \leq s_0^k < \alpha_0 < s_0^l \leq b_0,$$

hence $p_0 = s_0^k < s_0^l$ and hence contradicting the fact that $p_0 = r_0^l + s_0^l$. Therefore there is no $p \in T_i^* \cap T_j^*$ and hence $T_i^* \cap T_j^* = \oslash$. □

Using Proposition 6.6 (i), (iii) and (ii), it follows that

$$|T_i| = \sum_{k \in \mathbb{Z}_2^i} |S_i^k|.$$

Using the above properties of T_i, the size of its intersection with $\Delta_{\mathcal{B}}$ can be estimated.

PROPOSITION 6.7: *Assuming a coordinate order* $(l_1, \ldots, l_d)$,

$$|T_i^* \cap \Delta_{\mathcal{B}}| = \frac{d!}{i!} \, |T_i| \prod_{j=i+1}^{d} b_{l_j}.$$

Proof: $T_i \subset \Delta_{\mathcal{B}}$, $\Delta_{\mathcal{B}} = \{\, p \mid 0 \leq p_i < i\, b_i\}$. Thus,

$$\left|\Delta_{\mathcal{B}(l_{i+1} \ldots l_d)}\right| = \frac{d!}{i!} \prod_{j=i+1}^{d} b_{l_j}.$$

□

Using Propositions 6.1, 6.6 and 6.7, an upper bound on the size of the support of the Dixon polynomial can be derived. This also gives an upper bound on the size of the Dixon matrix and in turn, the degree of the projection operator. Here is the main result:

THEOREM 6.4:

$$\left|\Delta_{\mathcal{B}} - \bigcup_{i=1}^{d} T_i^*\right| = d! \prod_{i=1}^{d} b_i - \sum_{i=1}^{d} \frac{d!}{i!} |T_i| \prod_{j=i+1}^{d} b_j.$$

And, an upper bound on the size of the Dixon polynomial is given as:

$$|\Delta_{\mathcal{A}}| \leq d! \prod_{i=1}^{d} b_i - \sum_{i=1}^{d} \left(\frac{d!}{i!} |T_i| \prod_{j=i+1}^{d} b_{l_j} \right). \tag{3}$$

7. Polynomial Systems for which Resultants can be Computed Exactly

The above inequality (3) can be used to identify a class of unmixed polynomial systems whose support is such that the matrices constructed using the Dixon formulation are exact, i.e., the size coincides with the BKK bound. In other words, cases where

$$|\Delta_{\mathcal{A}}| = d\,!\prod_{i=1}^{d} b_i - \sum_{i=1}^{d}\left(\frac{d!}{i!}\;|T_i|\prod_{j=i+1}^{d} b_j\right) \qquad \Longrightarrow \qquad |\Delta_{\mathcal{A}}| = d\,!\ \mathsf{Vol}(\mathcal{A}).$$

For supports for which (3) reduces to the exact equality, the Dixon formulation produces exact resultants. This result generalizes most known results about unmixed polynomial systems for which resultants can be computed exactly. These results include n-degree systems [Kapur and Saxena, 1996] and bivariate corner-cut supports [Zhang and Goldman, 2000, Chionh, 2001, Chtcherba and Kapur, 2002e].

For example, for a n-degree system, all sets $S_i = \oslash$, and hence $|T_i| = 0$. Thus, $|\Delta_{\mathcal{A}}| = |\Delta_{\mathcal{B}}|$ and $|\Delta_{\mathcal{A}}| = d\,!\ \prod_{i=1}^{d} b_i = d\,!\ \mathsf{Vol}(\mathcal{A})$.

Below, we give a generalization of the concept of a bivariate corner-cut support introduced in [Zhang and Goldman, 2000, Chionh, 2001] for which the Dixon based resultant methods compute the exact resultant.

7.1. Corner-Cut in d-Dimension

A support $\mathcal{A}$ is called *corner-cut* if and only if for every coordinate order $(l_1, \ldots, l_d)$, the following conditions are satisfied:

1. the projection $\mathcal{A}_{(l_1 \ldots l_{d-1})} = \mathcal{B}_{(l_1 \ldots l_{d-1})}$, and
2. for each $k \in \mathbb{Z}_2^d$, S_d^k (as defined by definition 6.2) is a d-dimensional rectangle.

Figure 6 shows an example of 3D corner-cut support. If we consider any variable order and drop the last coordinate, a rectangular support is obtained, i.e. $\mathcal{A}_{(1,2)} = \mathcal{B}_{(1,2)}$ for any order, satisfying the first condition.

At the same time, $\mathcal{B} - \mathcal{A}$ is composed of the union of rectangular regions, each appearing in some corner of $\mathcal{B}$. Each of those rectangles is S_d^k for various values of $k \in \mathbb{Z}_2^3$, thus satisfying the second condition.

In the bivariate case, it is always the case that $\mathcal{A}_{(l_1)} = \mathcal{B}_{(l_1)}$. Thus, the first condition is trivially true. The above definition of a corner-cut support is a generalization of a bivariate corner cut support introduced in [Zhang and Goldman, 2000, Chionh, 2001] and studied in [Chtcherba and Kapur, 2002e] to higher dimensions. In that sense, we have settled open problems posed in [Zhang and Goldman, 2000, Chionh, 2001].

Below, we prove that if a support is corner cut, the size of the Dixon polynomial and hence, the degree of the associated projection operator equals the BKK

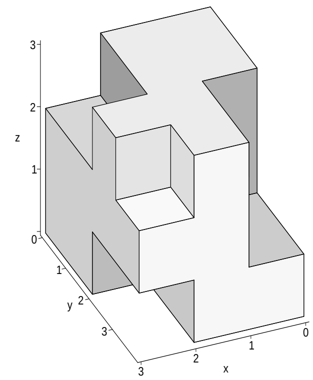

Figure 6: 3D corner cut support support.

bound which is also the degree of its toric resultant. Therefore, the projection operator extracted from the Dixon matrix is precisely the resultant of the given polynomial system.

Theorem 7.1 (d-dimensional Corner-Cut): *Given generic unmixed polynomial system with a* corner cut *support $\mathcal{A}$, the Dixon Matrix is exact for any variable order used to construct it.*

Proof: It will be proved that

$$d\,!\ \mathsf{Vol}(\mathcal{A}) = d\,!\prod_{i=1}^{d} b_i - \sum_{i=1}^{d}\left(\frac{d!}{i!}\ |T_i| \prod_{j=i+1}^{d} b_{l_j}\right) = d\,!\prod_{i=1}^{d} b_i - |T_d|,$$

The second equality above implies that $\mathcal{A}$ is corner cut in which $T_i = \oslash$ for all $1 \le i < d$; this implies that $|\Delta_{\mathcal{A}}| = d\,!\,\mathsf{Vol}(\mathcal{A})$. Also note because of the corner cut condition, the variable order does not change the upper bound.

For $k \in \mathbb{Z}_2^d$, let $b^k = (b'_1, \ldots, b'_d)$ where $b'_i = b_i$ if $k_i = 1$ and $b'_i = 0$ otherwise, that is, let b^k be the point in k^{th} corner of the bounding box. Consider the convex hull complement of $\mathcal{A}$ and its partition

$$Q = \mathsf{Conv}(\mathcal{B}) - \mathsf{Conv}(\mathcal{A}), \qquad \text{and} \qquad Q^k = \{\ q \mid q \in Q \quad \text{and line} \quad [b^k, q] \subset Q\ \}.$$

This set was also used for a bivariate corner cut support in [Chtcherba and Kapur, 2002e]; here we consider its generalization to the multivariate case. Because $\mathcal{A}$ is corner cut, note that for $k, l \in \mathbb{Z}_2^d$ and $k \ne l$,

$$Q^k \cup Q^l = \oslash \qquad \text{and} \qquad Q = \bigcup_{k \in \mathbb{Z}_2^d} Q^k.$$

Since

$$|T_d| = \sum_{k \in \mathbb{Z}_2^d} |S_d^k|,$$

it can be proved that

$$|S_d^k| = d\,!\ \mathsf{Vol}(Q^k),$$

from which the statement of the theorem follows.

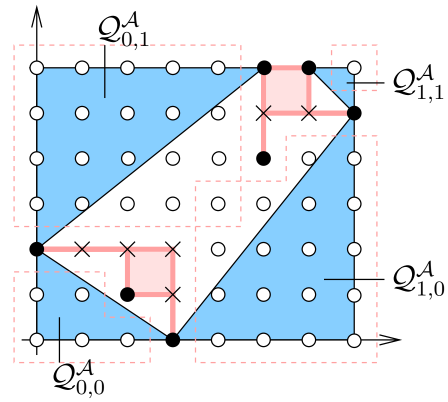

Figure 7: Newton polytope Complement.

Since each S_d^k is rectangular, the size of

$$|S_d^k| = \prod_{i=1}^{d} s_i,$$

where s_i is the number of points of S_d^k along the i^{th} coordinate.

But Q^k is a corner simplex whose sides are of length s_i; hence, its volume is

$$\mathsf{Vol}(Q^k) = \frac{1}{d\,!} \prod_{i=1}^{d} s_i;$$

therefore, $d\,!\ \mathsf{Vol}(Q^k) = |S_d^k|$. Thus,

$$\begin{aligned} d\,!\ \mathsf{Vol}(\mathcal{A}) &= d\,!\ \mathsf{Vol}(\mathcal{B}) - d\,!\ \mathsf{Vol}(Q) = |\Delta_{\mathcal{B}}| - \sum_{k \in \mathbb{Z}^d} d\,!\ \mathsf{Vol}(Q^k) \\ &= |\Delta_{\mathcal{B}}| - \sum_{k \in \mathbb{Z}^d} |S_d^k| = |\Delta_{\mathcal{B}}| - \sum_{k \in \mathbb{Z}^d} |T_d^k| = \left| \Delta_{\mathcal{B}} - \bigcup_{k \in \mathbb{Z}^d} T_d^k \right| \\ &= |\Delta_{\mathcal{B}} - T_d|\,. \end{aligned}$$

Since by Theorem 6.4, $d\,!\ \mathsf{Vol}(\mathcal{A}) \le |\Delta_{\mathcal{A}}| \le |\Delta_{\mathcal{B}} - T_d|$, it follows that $d\,!\ \mathsf{Vol}(\mathcal{A}) = |\Delta_{\mathcal{A}}|$, i.e., the Dixon matrix is exact. □

7.2. d-Dimensional Almost Corner-Cut Supports

A support $\mathcal{A}$ is called *almost corner-cut* if and only if the following conditions are satisfied:

1. There exist **a unique fixed** coordinate l_d, $1 \le l_d \le d$, (the corresponding variable is chosen to be substituted the last in the construction) such that for all coordinate orders $(l_1, \ldots, l_{d-1}, l_d)$, in which x_{l_d} is the last variable in the variable order, $\mathcal{A}_{(l_1 \ldots l_{d-1})} = \mathcal{B}_{(l_1 \ldots l_{d-1})}$.

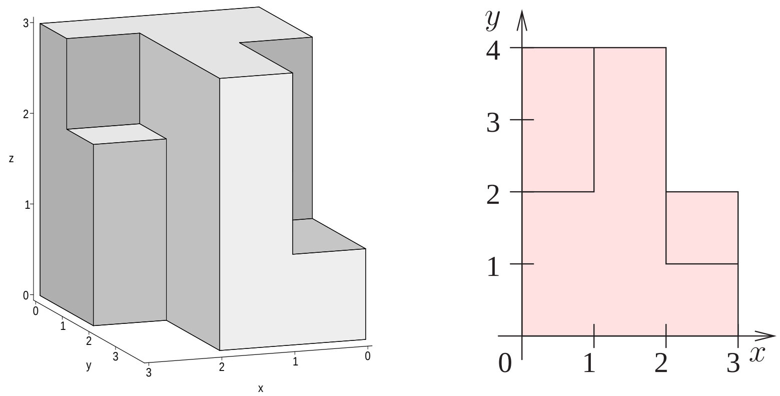

Figure 8: Almost corner cut support $\mathcal{A}$ and its projection $\mathcal{A}_{(1,2)}$.

2. for each $k \in \mathbb{Z}_2^d$, S_d^k is a d-dimensional rectangle.
3. for each $k \in \mathbb{Z}_2^{d-1}$, S_{d-1}^k is a $d-1$ dimensional rectangle, where the coordinate order is fixed as in condition (1).

For example, the support in Figure 8 is not corner-cut but is almost corner-cut. The example in Figure 4 for instance is neither corner cut nor almost corner-cut, as there are two choices for the last coordinate for which $\mathcal{A}_{(l_1 \dots l_{d-1})} \neq \mathcal{B}_{(l_1 \dots l_{d-1})}$.

To show that for an almost corner cut support, the Dixon formulation computes the resultant exactly, we use the same proof as above for a corner cut support.

Theorem 7.2: *Given a generic unmixed polynomial system with* almost corner-cut *support $\mathcal{A}$, the Dixon Matrix is exact for every variable order in which the last coordinate satisfies the above stated properties of in the definition of an almost corner cut support.*

Proof: It is shown below that the lower and upper bounds on $|\Delta_{\mathcal{A}}|$ are the same, that is,

$$d\,!\ \mathsf{Vol}(\mathcal{A}) = d\,!\prod_{i=1}^{d} b_i - \sum_{i=1}^{d} \left(\frac{d!}{i!}\ |T_i| \prod_{j=i+1}^{d} b_{l_j} \right).$$

Since $T_i = \oslash$ for $i < d-1$, it will be shown that

$$d\,!\ \mathsf{Vol}(\mathcal{A}) = d\,!\prod_{i=1}^{d} b_i - |T_d| - d\, b_{l_d}\, |T_{d-1}|.$$

Let Q and Q^k be as in the proof of Theorem 7.1. In this case,

$$Q = \bigcup_{k \in \mathbb{Z}_2^d} Q^k, \qquad \text{but not necessarily} \qquad Q^k \cap Q^l = \oslash,$$

for $k, l \in \mathbb{Z}_2^d$ and $k \neq l$.

If $Q^k \cap Q^l \neq \oslash$, then $k_i = l_i$ for all $i \neq d$. So let $k' = (k_1, \ldots, k_{d-1})$ and note that both b^k and b^l are in $S^{k'}_{(d-1,*)}$. In general,

$$\begin{aligned}\mathsf{Vol}(Q^k \cup Q^l) &= \big(\mathsf{Vol}(Q^k) - \mathsf{Vol}(Q^k \cap Q^l)\big) \\ &+ \big(\mathsf{Vol}(Q^l) - \mathsf{Vol}(Q^k \cap Q^l)\big) \\ &+ \mathsf{Vol}(Q^k \cap Q^l).\end{aligned}$$

but

$$d\,!\,\big(\mathsf{Vol}(Q^k) - \mathsf{Vol}(Q^k \cap Q^l)\big) = |S^k_{d-1}|$$

and

$$d\,!\,\big(\mathsf{Vol}(Q^l) - \mathsf{Vol}(Q^k \cap Q^l)\big) = |S^l_d|,$$

and also

$$d!\mathsf{Vol}(Q^k \cap Q^l) = b_{l_d}\,|S^{k'}_{d-1}|.$$

which can be verified in the same manner as in Theorem 7.1.

Hence, all the volume "missing" from $\mathcal{B}$ that is the volume of Q, equals $|T_d| + b_{l_d}\,|T_{d-1}|$, that is,

$$\begin{aligned}d\,!\,\mathsf{Vol}(\mathcal{A}) &= d\,!\,\mathsf{Vol}(\mathcal{B}) - d\,!\,\mathsf{Vol}(Q) \\ &= |\Delta_{\mathcal{B}}| - \sum_{k \in \mathbb{Z}^d} |S^k_d| - d \sum_{k' \in \mathbb{Z}^{d-1}} b_{l_d} |S^{k'}_d| \\ &= |\Delta_{\mathcal{B}}| - |T_d| - d\, b_{l_d}\,|T_{d-1}|.\end{aligned}$$

Since the upper and lower bounds for $|\Delta_{\mathcal{A}}|$ are the same, it follows that $d\,!\,\mathsf{Vol}(\mathcal{A}) = |\Delta_{\mathcal{A}}|$, implying that the Dixon-based methods compute the resultant in this case exactly. □

The main reason for the above argument not being applicable to the general case is that there does not exist one-to-one correspondence between S_i's and T_j's. Subsets T_j's depend on the projection chosen where as S_i's do not. A complete analysis must consider the dependence of T_j's on the variable order chosen.

We have thus settled an open problem raised in [Zhang and Goldman, 2000] of generalizing a bivariate corner cut support to the general multidimensional case. Corner-cut supports can perhaps be generalized in many different ways. It is proved above that the Dixon-based resultant methods compute resultants of generic unmixed polynomial systems exactly if their supports are corner-cut or almost corner-cut as defined above.

There are however families of generic unmixed polynomial systems whose support is neither corner-cut nor almost corner-cut, yet the Dixon formulation still produces exact resultants. A notable family of such polynomial systems is that of multi-graded systems introduced in [Morgan and Sommese, 1987] and analyzed for the Dixon construction in [Chtcherba and Kapur, 2000a]. Therefore there are other classes of supports for which the Dixon formulation is exact. So far the known classes are: multigraded, corner-cut and almost corner cut. It would be interesting to have an example which is not in one of the above classes.

8. Conclusions

The paper generalizes the results in [Zhang and Goldman, 2000, Chionh, 2001, Chtcherba and Kapur, 2002e] for the bivariate case to the general case. Using the concepts of support-interior points and support hull of the support of a generic unmixed multivariate polynomial system, the concept of a d-dimensional corner-cut support is defined. It is proved that for generic unmixed polynomial systems with d-dimensional corner-cut supports, the Dixon-based resultant methods (both the generalized Dixon method as defined in [Kapur et al., 1994] as well as the Dixon multiplier method defined in [Chtcherba and Kapur, 2000b, 2002a]) generate exact resultants. As a bye-product, the Dixon multiplier method also produces Sylvester-type resultant matrices for generic unmixed polynomial systems with d-dimensional corner-cut supports. Further, the variable ordering used in constructing Dixon-based resultant matrices does not affect the performance the result, i.e., (exact) resultants are generated irrespective of the variable ordering as well as the complexity of the method is not affected by the chosen variable ordering. This settles an open problem in [Zhang and Goldman, 2000].

It is also shown that these results can be generalized for generic unmixed polynomial systems with d-dimensional almost-corner-cut supports if certain variable orderings are chosen (these orderings only fix the last variable to be substituted in the construction).

A tighter bound on the size of the Dixon matrix and hence, on the degree of the projection operator extracted from it, is shown. This bound is based on analyzing how much a given support deviates from the support of an associated n-degree system for which the Dixon formulation is known to produce the exact resultant. This improves upon the related bounds proved in [Kapur and Saxena, 1996] and [Saxena, 1997]. As in the bivariate case, the size of the support of the Dixon polynomial of a given generic unmixed polynomial system is shown to be lower than or equal to that of an associated n-degree system minus the sum of all support hull complements. For a generic unmixed polynomial system whose support is either corner-cut or almost corner-cut, it is proved that the Dixon resultant formulation computes the resultant exactly (without producing any extraneous factors).

The above analysis also gives sharper bounds on the complexity of resultant computations based on the Dixon formulation in terms of its support since the complexity is governed by the determinant computations of Dixon matrices. Any deviation from an n-degree support is abstracted by notion of support complement, from which lower bound on deviation from the Dixon polynomial support of n-degree system is obtained and hence obtaining tighter upper bound on the size of the Dixon matrix.

The insight developed for defining almost-corner-cut supports is likely to be helpful in defining a heuristic for variable ordering for unmixed as well as mixed polynomial systems that is likely to lead to projection operators with extraneous factors of lower degrees. A method for finding translation vectors as well as term for constructing Dixon-based matrices is being investigated by generalizing the ideas developed in [Chtcherba and Kapur, 2002e] for the bivariate case.

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